数理解析研究所講究録
1126巻 2000年 1-13

IS THE FREE TOPOLOGICAL GROUP ON A C.C.C. SPACE C.C.C. ?

静岡大学教育学部　山田　耕三 (Kohzo Yamada)

ABSTRACT. Let $F(X)$ and $A(X)$ be respectively the free topological group and the free abelian topological group on a Tychonoff space X. For all natural number n we denote by $F_n(X)$ ($A_n(X)$) the subset of $F(X)$ ($A(X)$) consisting of all words of reduced length $\leq n$. Then $F(X)$ ($A(X)$) is the union of $\{F_n(X) : n \in \mathbb{N}\}$ ($\{A_n(X) : n \in \mathbb{N}\}$). In addition, for every $n \in \mathbb{N}$, $F_n(X)$ ($A_n(X)$) is a continuous image of $(X \oplus X^{-1} \oplus \{e\})^n$. Therefore, it follows that if we assume MA, then the free (abelian) topological group on a c.c.c. space is also c.c.c. On the other hand, we show here that if we assume the existence of a Suslin line, then there is a c.c.c. space X such that neither $F(X)$ nor $A(X)$ is c.c.c. This means that the question "Is the free (abelian) topological group on a c.c.c. space c.c.c. ?" is consistent with ZFC.

1 Introduction

The results in §3 of this note are joint work with Professor Gary Gruenhage (Auburn University).

All spaces are assumed to be Tychonoff. Let $F(X)$ and $A(X)$ be respectively the free topological group and the free abelian topological group on a Tychonoff space X in the sense of Markov [5]. For each $n \in \mathbb{N}$, $F_n(X)$ stands for a subset of $F(X)$ formed by all words whose reduced length is less than or equal to n. Then $F(X)$ is the union of $F_n(X)$, $n \in \mathbb{N}$. This concept is defined for $A(X)$ in the same fashion.

In this note, we consider the c.c.c. property of $F(X)$ and $A(X)$. Recall that a topological space X has the *countable chain condition* (c.c.c.) iff there is no uncountable family of pairwise disjoint non-empty open subsets of X. Since both $F_n(X)$ and $A_n(X)$ are respectively continuous images of $(X \oplus X^{-1} \oplus \{e\})^n$ and $(X \oplus -X \oplus \{0\})^n$ for each $n \in \mathbb{N}$, it is easy to show that if we assume MA, then both $F(X)$ and $A(X)$ on a c.c.c. space X is also c.c.c.

On the other hand, Tkačenko [7] proved the following.

Theorem 1.1 *If a space X is pseudocompact, then both $F(X)$ and $A(X)$ are c.c.c. spaces.*

Of course, the reverse implication of the above result does not hold. For, both $F(X)$ and $A(X)$ on a separable space X is separable. Thus, for example, $F(\mathbb{R})$ and $A(\mathbb{R})$ are c.c.c. The Tkačenko's result asserts that $F(\omega_1)$ and $A(\omega_1)$ are c.c.c. however the ordinal space ω_1 is not c.c.c. Since $F(X)$ and $A(X)$ on a c.c.c. space X are c.c.c. if we assume MA, one might conjecture that if the finite product of a space X, in particular X^2, has an uncountable pairwise disjoint family of non-empty open subsets, then neither $F(X)$ nor $A(X)$ could be c.c.c. However, by the above Tkačenko's result, the free (abelian) topological group on a compact Suslin line is c.c.c. and also since there is a pseudocompact space X, in ZFC, whose square has an uncountable discrete family of non-empty open subsets (apply Example 3.10.19 in [2]), both $F(X)$ and $A(X)$ on the space X are c.c.c. Of course, if we assume more strongly that a space X has an uncountable discrete family of non-empty open subsets, then we can prove that neither $F(X)$ nor $A(X)$ is c.c.c. applying the Graev's continuous pseudometric on $F(X)$ $(A(X))$ (see Theorem 1 of [3]). However, we should mention that $F(X)$ and $A(X)$ are not necessary to be c.c.c. on a space X which does not have an uncountable discrete family of non-empty open subsets. For example, let X be the one-point Lindelöfication of an uncountable discrete space. Then X is a P-space, and hence so is $F(X)$ $(A(X))$, and the pseudocharacter of $F(X)$ $(A(X))$ is uncountable. It follows that $F(X)$ $(A(X))$ is not c.c.c.

The above results suggest that it is not so easy to clarify the question whether both $F(X)$ and $A(X)$ on a c.c.c. space X are c.c.c. or not. In addition, we don't know whether the free (abelian) topological group on a (of course, non-compact) Suslin line is c.c.c. or not. However, Gruenhage and the author could recently construct a c.c.c. space X such that neither $F(X)$ and $A(X)$ is c.c.c. under the assumption of the existence of a Suslin line. Therefore, we can show at least that the statement "the free (abelian) topological group on a c.c.c. space is c.c.c." is consistent with ZFC.

In the next section, we introduce neighborhood of the identity in $F(X)$ and $A(X)$, respectively, and which are used for proving the results in §3. By $\mathbb{N}$ we denote the set of all natural numbers. We refer to [4] for elementary properties of topological groups and to [1] and [3] for the main properties of free topological groups.

2 Neighborhoods of the identity in $F(X)$ and $A(X)$

We first introduce the neighborhoods of 0 in $A(X)$ constructed by Tkačenko [8] and Pestov [6].

Let $\mathcal{U}_X$ be the universal uniformity on a space X. For each $P = \{U_1, U_2, \dots\} \in (\mathcal{U}_X)^\omega$, let

$$V(P) = \{x_1 - y_1 + x_2 - y_2 + \cdots + x_k - y_k : (x_i, y_i) \in U_i \text{ for } i = 1, \dots, k, k \in \mathbb{N}\},$$

and $\mathcal{V} = \{V(P) : P \in (\mathcal{U}_X)^\omega\}$. Then the following is known.

Theorem 2.1 **([8],[6])** *For a space X, $\mathcal{V}$ is a neighborhood base at* 0 *in $A(X)$.*

In the non-abelian case, we introduce the neighborhoods of e in $F(X)$ which are defined by the author [10].

Let X be a space and $\overline{X} = X \oplus \{e\} \oplus X^{-1}$, where e is the identity of $F(X)$. Fix an arbitrary $n \in \mathbb{N}$. For a subset U of $\overline{X}^2$ which includes the diagonal of $\overline{X}^2$, let $G_n(U)$ be a subset of $F_{2n}(X)$ which consists of the identity e and all words g satisfying the following conditions;

(1) g can be represented as the reduced form $g = x_1 x_2 \cdots x_{2k}$, where $x_i \in \overline{X}$ for $i = 1, 2, \dots, k$ and $1 \le k \le n$,

(2) there is a partition $\{1, 2, \dots, 2k\} = \{i_1, i_2, \dots, i_k\} \cup \{j_1, j_2, \dots, j_k\}$,

(3) $i_1 < i_2 < \cdots < i_k$ and $i_s < j_s$ for $s = 1, 2, \dots, k$,

(4) $(x_{i_s}, x_{j_s}^{-1}) \in U$ for $s = 1, 2, \dots, k$ and

(5) $i_s < i_t < j_s \Longleftrightarrow i_s < j_t < j_s$ for $s, t = 1, 2, \dots, k$.

Then it was proved that $G_n(U)$ is a neighborhood of e in $F_{2n}(X)$ for every $U \in \mathcal{U}_X$ and $n \in \mathbb{N}$. To prove it, the author [10] applied the following fact:

From Graev's construction of $\hat{d}$ which is a continuous pseudometric on $F(X)$ extending a continuous pseudometric d on X (see the proof of Theorem 1 in [3]), there exists a partition $\{1, 2, \ldots, 2n\} = \{i_1, i_2, \ldots, i_k\} \cup \{j_1, j_2, \ldots, j_k\}$ satisfying (3) and (5) of the definition of $G_n(U)$ and that $\hat{d}(g, e) = \sum_{p=1}^{k} \overline{d}(x_{i_p}, y_{j_p}^{-1})$.

However, Graev didn't describe the existence of the above partition in the proof of Theorem 1 in [3], and also there are no papers in which the above fact is mentioned with its proof. So, we give here another proof. Though it is rather long and complicated, the reader will be sure that each $G_n(U)$ is a neighborhood of e in $F_{2n}(X)$.

We introduce the following neighborhood base at e in $F(X)$ constructed by Tkačenko [9] which are used in the proof. Let X be a space. For each $n \in \mathbb{N}$, we define a mapping j_n from $\overline{X}^n \times \overline{X}^n$ to $F_{2n}(X)$ by $j_n((\boldsymbol{x}, \boldsymbol{y})) = i_n(\boldsymbol{x}) i_n(\boldsymbol{y})^{-1}$ for every $(\boldsymbol{x}, \boldsymbol{y}) \in \overline{X}^n \times \overline{X}^n$. Let $\mathcal{U}_n$ be the universal uniformity on $\overline{X}^n$ for each $n \in \mathbb{N}$. For each $R = \{U_n : n \in \mathbb{N}\} \in \prod_{i=1}^{\infty} \mathcal{U}_n$, we put

$$W_n(R) = \bigcup \{ j_{\pi(1)}(U_{\pi(1)}) \cdots j_{\pi(n)}(U_{\pi(n)}) : \pi \in S_n \} \text{ and } W(R) = \bigcup_{n=1}^{\infty} W_n(R),$$

where S_n is the permutation group on $\{1, 2, \ldots, n\}$. Then Tkačenko [9] proved that $\{W(R) : R \in \prod_{i=1}^{\infty} \mathcal{U}_n\}$ is a neighborhood base at e in $F(X)$.

Let $\mathcal{U}$ be a uniformity on a space X and $U \in \mathcal{U}$. The set $U \circ U \in \mathcal{U}$ is defined as follows: $(x, z) \in U \circ U$ iff there is $y \in X$ such that $(x, y) \in U$ and $(y, z) \in U$. The following technical lemma is also used in the proof.

Lemma 2.2 ([8]) *Let $\mathcal{U}$ be a uniformity on a space X and $\{U_0, U_1, \ldots\}$ be a sequence in $\mathcal{U}$ such that $U_{n+1} \circ U_{n+1} \circ U_{n+1} \subseteq U_n$ for each $n = 0, 1, \ldots$. Then for each $k \in \mathbb{N} \cup \{0\}$ and $k_1, k_2, \ldots, k_p \in \mathbb{N} (p \in \mathbb{N})$ such that $\sum_{i=1}^{p} 2^{-k_i} < 2^{-k}$,*

$$U_{k_1} \circ U_{k_2} \circ \cdots \circ U_{k_p} \subseteq U_k.$$

We define some notation that is used in the proof. Let $x_1 x_2 \cdots x_n$ be a form, where $n \in \mathbb{N}$ and $x_i \in \overline{X}$ for each $i \leq n$. For $a, b \in \{x_i : i = 1, 2, \ldots, n\}$, the inequality $a \leq b$ means that $a = b$ or a appears on the left of b in the above form, that is, there are $i, j \in \{1, 2, \ldots, n\}$ such that $i \leq j$, $a = x_i$ and $b = x_j$. The symbols $[x_i, x_j]$ and (x_i, x_j) means the forms $x_i x_{i+1} \cdots x_j$ and $x_{i+1} \cdots x_{j-1}$, respectively. For $a, b, c, d \in \{x_i : i = 1, 2, \ldots, n\}$, the symbol $[a, b] \subseteq [c, d]$ means that $c \leq a \leq b \leq d$.

Theorem 2.3 *Let X be a space. Then for every $U \in \mathcal{U}_X$ and every $n \in \mathbb{N}$, $G_n(U)$ is a neighborhood of e in $F_{2n}(X)$.*

Proof. Let $U \in \mathcal{U}_{\overline{X}}$. Then there is a sequence $\{U_m \in \mathcal{U}_{\overline{X}} : m \in \mathbb{N}\}$ such that $U_0 = U$, $U_m \subseteq X^2 \oplus (X^{-1})^2 \oplus \{(e,e)\}$, $U_m = U_m^{-1}$ and $\{(x^{-1}, y^{-1}) : (x,y) \in U_m \cap X^2\} = U_m \cap (X^{-1})^2$ for $m = 1, 2, \dots$ and $U_{m+1} \circ U_{m+1} \circ U_{m+1} \subseteq U_m$ for $m = 0, 1, \dots$ Since every $\mathcal{U}_k$ is the universal uniformity, we can take $V_k \in \mathcal{U}_k$ such that $V_k \subseteq (U_{\frac{k(k+1)}{2}})^k$ for each $k \in \mathbb{N}$. If we put $R = \{V_1, V_2, \dots\}$, then $R \in \prod_{i=1}^{\infty} \mathcal{U}_i$. To prove that $G_n(U)$ is a neighborhood of e in $F_{2n}(X)$, we shall show here that $W(R) \cap F_{2n}(X) \subseteq G_n(U)$.

Let $g \in W(R) \cap F_{2n}(X)$. Then, by the definition of $W(R)$, there are $k \in \mathbb{N}$, $\pi \in S_k$ and $(\boldsymbol{x}_{\pi(i)}, \boldsymbol{y}_{\pi(i)}) \in V_{\pi(i)}$ for $i = 1, 2, \dots, k$ such that

$$g = j_{\pi(1)}((\boldsymbol{x}_{\pi(1)}, \boldsymbol{y}_{\pi(1)})) \cdot j_{\pi(2)}((\boldsymbol{x}_{\pi(2)}, \boldsymbol{y}_{\pi(2)})) \cdot \dots \cdot j_{\pi(k)}((\boldsymbol{x}_{\pi(k)}, \boldsymbol{y}_{\pi(k)})).$$

For convenience, we only prove when π is the identity (it can be proved similarly in the general case). So we can put $g = \boldsymbol{x}_1 \boldsymbol{y}_1^{-1} \boldsymbol{x}_2 \boldsymbol{y}_2^{-1} \cdots \boldsymbol{x}_k \boldsymbol{y}_k^{-1}$. For each $i = 1, 2, \dots, k$ let $\boldsymbol{x}_i = (x_{p_i+1}, x_{p_i+2}, \dots, x_{p_i+i})$ and $\boldsymbol{y} = (y_{p_i+1}, y_{p_i+2}, \dots, y_{p_i+i})$, where $p_i = \dfrac{i(i-1)}{2}$ and $x_j, y_j \in \overline{X}$ for $j = p_i + 1, p_i + 2, \dots p_i + i$. Then, g is represented as follows;

$$\begin{aligned} g = x_1 y_1^{-1} x_2 x_3 y_3^{-1} y_2^{-1} \cdots x_{p_i+1} x_{p_i+2} \cdots x_{p_i+i} \, y_{p_i+i}^{-1} \cdots y_{p_i+2}^{-1} \, y_{p_i+1}^{-1} \cdots \\ x_{p_k+1} x_{p_k+2} \cdots x_{p_k+k} \, y_{p_k+k}^{-1} \cdots y_{p_k+2}^{-1} \, y_{p_k+1}^{-1}. \quad (1) \end{aligned}$$

For each $i = 1, 2, \dots, k$, since $(\boldsymbol{x}_i, \boldsymbol{y}_i) \in V_i \subseteq (U_{\frac{i(i+1)}{2}})^i \subseteq U_{\frac{i(i-1)}{2}+1} \times U_{\frac{i(i-1)}{2}+2} \times \cdots \times U_{\frac{i(i-1)}{2}+i} = U_{p_i+1} \times U_{p_i+2} \times \cdots \times U_{p_i+i}$, we have that $(x_{p_i+j}, y_{p_i+j}) \in U_{p_i+j}$ for each $j = 1, 2, \dots, i$. This means that

$$(x_l, y_l) \in U_l \text{ for every } l = 1, 2, \dots, p_k + k. \quad (2)$$

Put $A = \{x_l : l = 1, 2, \dots, p_k + k\} \cup \{y_l^{-1} : l = 1, 2, \dots, p_k + k\}$ and we assume that each element of A is distinct. Since $g \in F_{2n}(X)$ and the number of the elements of A is even, the reduced form of g can be represented as follows:

$$g = z_1 z_2 \cdots z_{2q}, \text{ where } q \leq n \text{ and } z_i \in A \text{ for } i = 1, 2, \dots, 2q. \quad (3)$$

Since the length of $g \leq 2n$, the form (1) of g may contains letters that can be reduced. Now we fix a way to reduce g from the form (1) to the reduced form (3). To prove that $g \in G_n(U)$, we decompose the set $\{1, 2, \dots, 2q\}$ into two sets $\{i_1, i_2, \dots, i_k\}$ and $\{j_1, j_2, \dots, j_k\}$ in the following way.

Let $i_1 = 1$ and $a_{(i_1,1)} \in A$ such that $z_{i_1} = a_{(i_1,1)}$. Then there is $b_{(i_1,1)} \in A$ such that $\{a_{(i_1,1)}, b_{(i_1,1)}\} = \{x_l, y_l^{-1}\}$ for some $l = 1, 2, \dots, p_k + k$. If $b_{(i_1,1)}$ is an irreducible letter, then it is equal to some $z_i, i = 1, 2, \dots, 2q$. Let j_1 be such that $z_{j_1} = b_{(i_1,1)}$. Clearly $i_1 \neq j_1$. Otherwise, there is $a_{(i_1,2)} \in A$ such that $b_{(i_1,1)}$ is reduced by $a_{(i_1,2)}$. Again take $b_{(i_1,2)} \in A$ such that $\{a_{(i_1,2)}, b_{(i_1,2)}\} = \{x_l, y_l^{-1}\}$ for some $l = 1, 2, \dots, p_k + k$. It is clear that the elements $a_{(i_1,1)}, b_{(i_1,1)}, a_{(i_1,2)}$ and $b_{(i_1,2)}$ are distinct elements of A. If $b_{(i_1,2)}$ is an irreducible letter, then we can find a number $j_1 \in \{1, 2, \dots, 2q\} \setminus \{i_1\}$ with $z_{j_1} = b_{(i_1,2)}$. Otherwise, let $a_{(i_1,3)} \in A$ such that $b_{(i_1,2)}$ is reduced by $a_{(i_1,3)}$. Continue the step till we reach an irreducible letter $b_{(i_1,j)}$ and let j_1 be a number with $z_{j_1} = b_{(i_1,j)}$. Clearly it always exists, since A is a finite set.

For the sake of understanding we prepare here the following example:
Let $c_1, c_2, \dots, c_{12} \in X$ such that $(c_2, c_3^{-1}) \in V_1$, $((c_3^{-1}, c_2), (c_4^{-1}, c_1)) \in V_2$, $((c_5^{-1}, c_5, c_9)(c_6^{-1}, c_4, c_9)) \in V_3$ and $((c_6^{-1}, c_8, c_{10}, c_{11})(c_7^{-1}, c_7, c_{12}^{-1}, c_{11})) \in V_4$. Then,

$$
\begin{array}{ccccccccccccccccccccc}
 & & & & & z_{i_1} & & & & & & & & & & & & & & & \\
 & & & & & \Vert & & & & & & & & & & & & & & & \\
 & a_{(i_1,2)} & & a_{(i_1,3)} & & a_{(i_1,1)} & & a_{(i_1,5)} & & & & a_{(i_1,4)} & & a_{(i_1,6)} & & & & & & a_{(i_1,7)} & \\
 & \Vert & & \Vert & & \Vert & & \Vert & & & & \Vert & & \Vert & & & & & & \Vert & \\
g = & c_2^{-1} & c_3 & c_3^{-1} & c_2 & c_1^{-1} & c_4 & c_5^{-1} & c_5 & c_9 & c_9^{-1} & c_4^{-1} & c_6 & c_6^{-1} & c_8 & c_{10} & c_{11} & c_{11}^{-1} & c_{12} & c_7^{-1} & c_7 \\
 & & \Vert & & \Vert & & \Vert & & \Vert & & & & \Vert & & \Vert & & & & & & \Vert \\
 & & b_{(i_1,2)} & & b_{(i_1,1)} & & b_{(i_1,3)} & & b_{(i_1,4)} & & & & b_{(i_1,5)} & & b_{(i_1,7)} & & & & & & b_{(i_1,6)} \\
 & & & & & & & & & & & & & & \Vert & & & & & & \\
 & & & & & & & & & & & & & & z_{j_1} & & & & & &
\end{array}
$$

$$= c_1^{-1} \; c_8 \; c_{10} \; c_{12}.$$

Next let $i_2 = \min\{l : l \in \{1, 2, \dots, 2q\} \setminus \{i_1, j_1\}\}$. We start the same step from z_{i_2} and find z_{j_2}. Consequently we can get the sets $\{i_1, i_2, \dots, i_k\}$ and $\{j_1, j_2, \dots, j_k\}$ satisfying the following properties:

(4-1) $\{1, 2, \dots, 2q\} = \{i_1, i_2, \dots, i_q\} \cup \{j_1, j_2, \dots, j_q\}, \quad i_1 < i_2 < \dots < i_q,$

(4-2) $\{i_1, i_2, \dots, i_q, j_1, j_2, \dots, j_q\}$ consists of distinct numbers, and

for each $s \in \{1, 2, \dots, q\}$ there is a subset $A_s = \{a_{(i_s,1)}, b_{(i_s,1)}, a_{(i_s,2)}, b_{(i_s,2)}, \dots, a_{(i_s,u_s)}, b_{(i_s,u_s)}\}$ of A such that

(4-3) $z_{i_s} = a_{(i_s,1)}, z_{j_s} = b_{(i_s,u_s)}$ and $a_{(i_s,1)} < b_{(i_s,u_s)}$,

(4-4) $\{a_{(i_s,j)}, b_{(i_s,j)}\} = \{x_l, y_l^{-1}\}$ for some $l = 1, 2, \dots, p_k + k$, $j = 1, 2, \dots, u_s$,

(4-5) $b_{(i_s,j)}$ is reduced by $a_{(i_s,j+1)}$, i.e. $a_{(i_s,j+1)}^{-1} = b_{(i_s,j)}$ for $j = 1, 2, \dots, u_s - 1$,

(4-6) $\{a_{(i_s,j)}, b_{(i_s,j)} : j = 1, 2, \dots, u_s, s = 1, 2, \dots, q\}$ consists of distinct elements of A.

Let $s, r \in \{1, 2, \dots, q\}$, $t \in \{1, 2, \dots, u_s\}$ and $v \in \{1, 2, \dots, u_r\}$. By (4-4) there are $i, i' \in \{1, 2, \dots, i\}$ and $j, j' \in \{1, 2, \dots, i\}$ such that $\{a_{(i_s,t)}, b_{(i_s,t)}\} = \{x_{p_i+j}, y_{p_i+j}^{-1}\}$ and $\{a_{(i_r,v)}, b_{(i_r,v)}\} = \{x_{p_{i'}+j'}, y_{p_{i'}+j'}^{-1}\}$. If $a_{(i_s,t)} < a_{(i_r,v)} < b_{(i_s,t)}$, then since these letters appear in $\boldsymbol{x}_i \boldsymbol{y}_i^{-1}$, $b_{(i_r,v)}$ also appears between $a_{(i_s,t)}$ and $b_{(i_s,t)}$. On the other hand, assume that $t, v \geq 2$ and $b_{(i_s,t-1)} < b_{(i_r,v-1)} < a_{(i_s,t)}$. Since $b_{(i_s,t-1)}$ is reduced by $a_{(i_s,t)}$ by (4-5), $[b_{(i_s,t-1)}, a_{(i_s,t)}] = e$. Thus, each letter between $b_{(i_s,t-1)}$ and $a_{(i_s,t)}$ must be reduced by another letter between them. It follows that $b_{(i_s,t-1)} < a_{(i_r,v)} < a_{(i_s,t)}$. These arguments yield the following properties:
For each $s, r \in \{1, 2, \dots, q\}$, $t \in \{1, 2, \dots, u_s\}$ and $v \in \{1, 2, \dots, u_r\}$

$$\begin{aligned} a_{(i_s,t)} < a_{(i_r,v)} < b_{(i_s,t)} &\iff a_{(i_s,t)} < b_{(i_r,v)} < b_{(i_s,t)} \text{ and} \\ b_{(i_s,t)} < a_{(i_r,v)} < a_{(i_s,t)} &\iff b_{(i_s,t)} < b_{(i_r,v)} < a_{(i_s,t)}. \end{aligned} \tag{5}$$

For each $s, r \in \{1, 2, \dots, q\}$, $t \in \{2, 3, \dots, u_s\}$ and $v \in \{2, 3, \dots, u_r\}$

$$\begin{aligned} b_{(i_s,t-1)} < b_{(i_r,v-1)} < a_{(i_s,t)} &\iff b_{(i_s,t-1)} < a_{(i_r,v)} < a_{(i_s,t)} \text{ and} \\ a_{(i_s,t)} < b_{(i_r,v-1)} < b_{(i_s,t-1)} &\iff a_{(i_s,t)} < a_{(i_r,v)} < b_{(i_s,t-1)}. \end{aligned} \tag{6}$$

By (4-1),(4-2) and (4-3), to show that $g \in W_n(U)$ it suffices to prove the following two claims.

Claim 1. $(z_{i_s}, z_{j_s}^{-1}) \in U$ for each $s = 1, 2, \dots, q$.

For each $j = 1, 2, \dots, u_s$, by (4-4), we can choose a number $l(i_s, j) \in \{1, 2, \dots, p_k + k\}$ such that $\{a_{(i_s,j)}, b_{(i_s,j)}\} = \{x_{l(i_s,j)}, y_{l(i_s,j)}^{-1}\}$. From the forms of $U_m, m \in \mathbb{N}$ and the properties of (2), (4-3) and (4-5), we have that $(z_{i_s}, z_{j_s}^{-1}) = (a_{(i_s,1)}, b_{(i_s,u_s)}^{-1}) \in U_{l(i_s,1)} \circ U_{l(i_s,2)} \circ \cdots \circ U_{l(i_s,u_s)}$. If we put $l_s = \min\{l(i_s, j) : j = 1, 2, \ldots, u_s\}$, then $l_s \geq 1$. Therefore, by Lemma 5.1, we conclude that $(z_{i_s}, z_{j_s}^{-1}) \in U_{l(s)-1} \subseteq U_0 = U$.

Claim 2. $i_s < i_r < j_s \iff i_s < j_r < j_s$ for each $s, r \in \{1, 2, \ldots, q\}$.

Fix $s, r \in \{1, 2, \ldots, q\}$. We shall prove that if $a_{(i_s,1)} < a_{(i_r,1)} < b_{(i_s,u_s)}$, then $a_{(i_s,1)} < b_{(i_r,u_r)} < b_{(i_s,u_s)}$. Define a mapping ϕ from A to $\{1, 2, \ldots, k(k+1)\}$ by $\phi(x_{p_i+j}) = i(i-1)+j$ and $\phi(y_{p_i+j}^{-1}) = i(i+1)-j+1$ for each $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, i$. Then ϕ preserves the order, i.e. for $a, a' \in A$ if $a < a'$ (in the form (1)), then $\phi(a) < \phi(a')$. For each $t = 1, 2, \ldots, u_s$ let $\alpha(s,t)$ and $\beta(s,t)$ be the points of the plane $\mathbb{R}^2$ such that $\alpha(s,t) = (\phi(a_{(i_s,t)}), 0)$ and $\beta(s,t) = (\phi(b_{(i_s,t)}), 0)$. We define paths $P(s,t)$ from $\alpha(s,t)$ to $\beta(s,t)$ in $\mathbb{R}^2$ for $t = 1, 2, \ldots, u_s$, paths $Q(s, t-1)$ from $\beta(s, t-1)$ to $\alpha(s,t)$ in $\mathbb{R}^2$ for $t = 2, 3, \ldots, u_s$ and a path $Q(s, u_s)$ from $\beta(s, u_s)$ to $\alpha(s, 1)$, as follows:

(i) If there are $t_1, t_2, \ldots, t_r \in \{1, 2, \ldots, u_s\}$ such that $[c_{(i_s,t_j)}, d_{(i_s,t_j)}] \subsetneq [c_{(i_s,t)}, d_{(i_s,t)}]$ for each $j = 1, 2, \ldots, r$, where $c_{(i_s,j)} = \min\{a_{(i_s,j)}, b_{(i_s,j)}\}$ and $d_{(i_s,j)} = \max\{a_{(i_s,j)}, b_{(i_s,j)}\}$ for $j = t_1, t_2, \ldots, t_r, t$, then put

$$\begin{aligned} P(s,t) = \{(\phi(c_{(i_s,t)}), y) : 0 \leq y \leq r+1\} \cup \{(x, r+1) : \phi(c_{(i_s,t)}) \leq x \leq \phi(d_{(i_s,t)})\} \\ \cup \{(\phi(d_{(i_s,t)}), y) : 0 \leq y \leq r+1\}. \end{aligned}$$

(ii) If there are $t_1, t_2, \ldots, t_r \in \{1, 2, \ldots, u_s\}$ such that $[e_{(i_s,t_j)}, f_{(i_s,t_j)}] \subsetneq [e_{(i_s,t)}, f_{(i_s,t)}]$ for each $j = 1, 2, \ldots, r$, where $e_{(i_s,t_j)} = \min\{a_{(i_s,j)}, b_{(i_s,j-1)}\}$ and $f_{(i_s,j)} = \max\{a_{(i_s,j)}, b_{(i_s,j-1)}\}$ for $j = t_1, t_2, \ldots, t_r, t$, then put

$$\begin{aligned} Q(s,t) = \{(\phi(e_{(i_s,t)}), y) : -(r+1) \leq y \leq 0\} \\ \cup \{(x, -(r+1)) : \phi(e_{(i_s,t)}) \leq x \leq \phi(f_{(i_s,t)})\} \\ \cup \{(\phi(f_{(i_s,t)}), y) : -(r+1) \leq y \leq 0\}. \end{aligned}$$

(iii) If there are $t_1, t_2, \ldots, t_r \in \{1, 2, \ldots, u_s\}$ such that $[e_{(i_s,t_j)}, f_{(i_s,t_j)}] \subsetneq [a_{(i_s,1)}, b_{(i_s,u_s)}]$

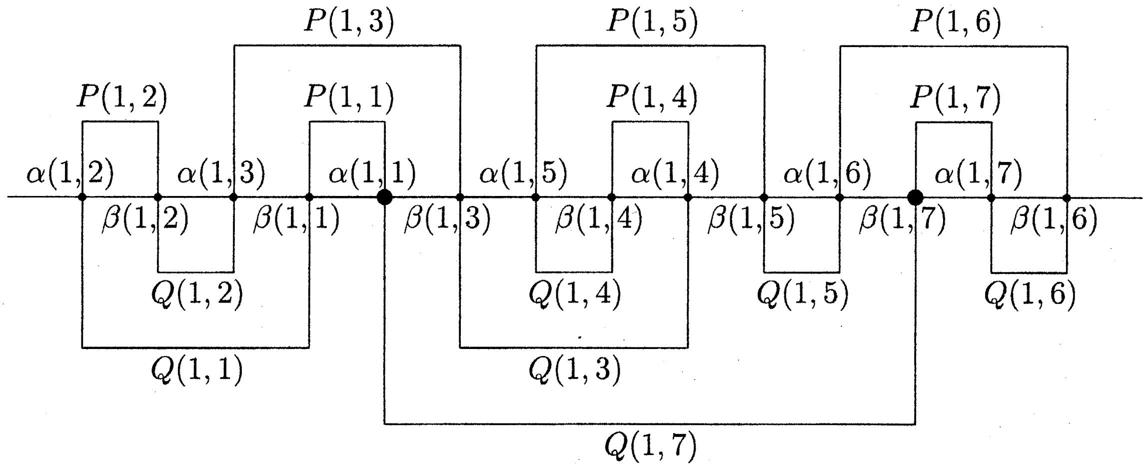
for each $j = 1, 2, \ldots, r$, then put

$$Q(s, u_s) = \{(\phi(a_{(i_s,1)}), y) : -(r+1) \le y \le 0\} \\ \cup \{(x, -(r+1)) : \phi(a_{(i_s,1)}) \le x \le \phi(b_{(i_s,u_s)})\} \\ \cup \{(\phi(b_{(i_s,u_s)}), y) : -(r+1) \le y \le 0\}.$$

Recall the example on page 5, that is, let

$$g = c_2^{-1}c_3c_3^{-1}c_2c_1^{-1}c_4c_5^{-1}c_5c_9c_9^{-1}c_4^{-1}c_6c_6^{-1}c_8c_{10}c_{11}c_{11}^{-1}c_{12}c_7^{-1}c_7 \\ = c_1^{-1}c_8c_{10}c_{12}$$

The following figure illustrates the points $\alpha(1,t) = (\phi(a_{(i_1,t)}), 0)$, $\beta(1,t) = (\phi(b_{(i_1,t)}), 0)$ and the paths $P(1,t)$, $Q(1,t)$, $t = 1, 2, \ldots, 7$ with respect the above word g.



The above constructions of the paths and the properties (5) and (6) say that $L = P(s,1) \cup Q(s,1) \cup P(s,2) \cup Q(s,2) \cup \cdots \cup P(s,u_s) \cup Q(s,u_s)$ is a simple closed curve in $\mathbb{R}^2$. Let $L(i_r) = \{(\phi(a_{(i_r,1)}), y) : y \le 0\}$. Since $z_{i_r} = a_{(i_r,1)}$ is irreducible, the letter $a_{(i_r,1)}$ cannot appear between $a_{(i_s,t)}$ and $b_{(i_s,t-1)}$ for each $t = 1, 2, \ldots, u_s$. Hence the half line $L(i_r)$ cannot intersect $Q(s,t)$ for each $t = 1, 2, \ldots, u_s - 1$. On the other hand, since $a_{(i_s,1)} < a_{(i_r,1)} < b_{(i_s,u_s)}$, $L(i_r)$ must intersect $Q(s,u_s)$. This means that the half line $L(i_r)$ intersects the simple closed curve L only once. Furthermore, the part of

$L(i_r)$ whose second coordinate is less than the one of $Q(s, u_s)$ is an unbounded set in $\mathbb{R}^2$. Therefore, these facts follow that the point $(\phi(a_{(i_r,1)}), 0)$ is inside of L. By the properties (5) and (6), we can construct a path from $(\phi(a_{(i_r,1)}), 0)$ to $(\phi(b_{i_r,u_r})), 0)$ that does not intersect L. So, the point $(\phi(b_{(i_r,u_r)}), 0)$ is also inside of L. Now suppose that $b_{(i_r,u_r)} < a_{(i_s,1)}$ or $b_{(i_s,u_s)} < b_{(i_r,u_r)}$. Then the half line $M(i_r) = \{(\phi(b_{(i_r,u_r)}), y) : y \leq 0\}$ cannot intersect $Q(s, u_s)$. Furthermore, since $a_{j_r} = b_{(i_r,u_r)}$ is irreducible, $b_{(i_r,u_r)}$ cannot appear between $a_{(i_s,t)}$ and $b_{(i_s,t-1)}$ for each $t = 2, 3, \ldots, u_s$, and hence $M(i_r)$ does not intersect any $Q(s, t)$. Therefore, the half line $M(i_r)$ does not intersect the simple closed curve L. Since $M(i_r)$ is an unbounded subset of $\mathbb{R}^2$, it follows that $M(i_r)$ is contained in the outside area of L, and hence so is $(\phi(b_{(i_r,u_r)}), 0)$. This is a contradiction.

By Claim 1 and 2, we can conclude that $g \in G_n(U)$. Therefore, it follows that $W(R) \cap F_{2n}(X) \subseteq G_n(U)$. □

For every $U \in \mathcal{U}_X$, let $G(U) = \bigcup_{n=1}^{\infty} G_n(U)$. Then, by Theorem 2.3, we have the following.

Theorem 2.4 *Let X be a space. Then for every $U \in \mathcal{U}_X$, $G(U)$ is a neighborhood of e in $F(X)$.*

3 Example

In this section, we shall construct a c.c.c. space X such that neither $F(X)$ nor $A(X)$ is c.c.c. under the assumption the existence of a Suslin line.

Let T be a Suslin tree such that each node has 2 immediate successors and X be the space of all branches which topology is induced by $\{[t] : t \in T\}$ as a clopen base, where $[t]$ means the set of all branches going through t. Then the space X is the required space. Since it is easy to show that X is a c.c.c. space, we need to prove that neither $F(X)$ nor $A(X)$ is c.c.c. To begin the proof, we start by defining some notations.

For every $x \in T$, the *height* of x in T, or $\mathrm{ht}\,(x, T)$, is $\mathrm{type}(\{y \in T : y < x\})$. Let $\alpha < \omega_1$ and $t_\alpha \in T$ such that $t_\alpha \in \mathrm{Lev}_\alpha(T)$, where $\mathrm{Lev}_\alpha(T) = \{x \in T : \mathrm{ht}\,(\{y \in T : y < x\}) = \alpha\}$. Since each node has 2 immediate successors, let t_α^0 and t_α^1 be the successors of t_α. Pick $b_\alpha^i \in [t_\alpha^i]$ and put $h_\alpha^i = \mathrm{type}\,(b_\alpha^i)$ for $i = 0, 1$. Then, there is a cofinal set W in ω_1 such that if $\alpha, \beta \in W$ and $\beta < \alpha$, then $h_\beta^i < \alpha$ for $i = 0, 1$. For every $\alpha < \omega_1$, let

$s_\beta(\alpha) < t_\alpha$ such that $s_\beta(\alpha) \in \mathrm{Lev}_\beta(T)$ and

$$\mathcal{P}_\alpha = \{[t^0_\alpha], [t^1_\alpha]\} \cup \{[s_\beta(\alpha)] \setminus [s_{\beta+1}(\alpha)] : \beta < \alpha\}.$$

Then $\mathcal{P}_\alpha$ is a partition of X. Since, for each $\beta < \alpha$, $[s_\beta(\alpha)] \setminus [s_{\beta+1}(\alpha)] = [s_\beta(\alpha)^i]$ for $i = 0$ or 1, $\mathcal{P}_\alpha$ is consisting of basic clopen subsets of X, and hence for every $P \in \mathcal{P}_\alpha$ and $P' \in \mathcal{P}_\beta$ P and P' are comparable, that is, if $P \cap P' \neq \emptyset$, then $P \subseteq P'$ or $P' \subseteq P$.

Let $\alpha \in W$. In the non-abelian case, put

$$U_\alpha = \bigcup\{P \times P : P \in \mathcal{P}_\alpha\} \cup \{(e, e)\} \cup \bigcup\{P^{-1} \times P^{-1} : P \in \mathcal{P}_\alpha\}.$$

Since the space X is paracompact and U_α is an open neighborhood of the diagonal $\Delta_{\overline{X}}$ in $\overline{X}^2$, we have that $U_\alpha \in \mathcal{U}_{\overline{X}}$. On the other hand, in the abelian case, put

$$V_\alpha = \bigcup\{P \times P : P \in \mathcal{P}_\alpha\}.$$

Then, we have that $V_\alpha \in \mathcal{U}_X$ by the same reason. In this note, we only prove that $F(X)$ is not c.c.c. In the abelian case, we can show that $A(X)$ is not c.c.c. with the similar argument if we use the neighborhoods $V(R_\alpha)$ of 0 in $A(X)$ instead of $G(U_\alpha)$, where $R_\alpha = \{V_\alpha, V_\alpha, \dots\} \in (\mathcal{U}_X)^\omega$.

We need the following technical lemmas.

Lemma 3.1 *Let $U \in \mathcal{U}_{\overline{X}}$. If $U = U^{-1}$, then $G(U) = G(U)^{-1}$.*

Lemma 3.2 *Let $\mathcal{A}$ and $\mathcal{B}$ be partitions of X such that every $A \in \mathcal{A}$ and $B \in \mathcal{B}$ are comparable. Put*

$$U = \bigcup\{A \times A : A \in \mathcal{A}\} \cup \{(e, e)\} \cup \bigcup\{A^{-1} \times A^{-1} : A \in \mathcal{A}\}$$
$$V = \bigcup\{B \times B : B \in \mathcal{B}\} \cup \{(e, e)\} \cup \bigcup\{B^{-1} \times B^{-1} : B \in \mathcal{B}\}.$$

Then $G(U) \cdot G(V) \subseteq G(U \cup V)$.

Theorem 3.3 *$F(X)$ is not c.c.c.*

Proof. Let $g_\alpha = b^0_\alpha b^1_\alpha$ for each $\alpha \in W$. Then each $g_\alpha \in F_2(X)$. To complete the proof, we shall prove that the family $\{g_\alpha G(U_\alpha) : \alpha \in W\}$ is pairwise disjoint. Suppose that $g_\alpha G(U_\alpha) \cap g_\beta G(U_\beta) \neq \emptyset$ for some $\alpha, \beta \in W$ with $\beta < \alpha$. Then, by Lemma 3.1 and 3.2,

$$g_\alpha^{-1} g_\beta \in G(U_\alpha) \cdot G(U_\beta)^{-1} = G(U_\alpha) \cdot G(U_\beta) \subseteq G(U_\alpha \cup U_\beta).$$

Since $g_\alpha^{-1} g_\beta = {b_\alpha^1}^{-1} {b_\alpha^0}^{-1} b_\beta^0 b_\beta^1$, by the definition of the neighborhood $G_2(U)$ of e in $F_4(X)$, the both of the pairs $({b_\alpha^1}^{-1}, {b_\beta^1}^{-1})$ and $({b_\alpha^0}^{-1}, {b_\beta^0}^{-1})$ must be in $U_\alpha \cup U_\beta$, and hence $(b_\alpha^i, b_\beta^i) \in U_\alpha \cup U_\beta$ for $i = 0, 1$. On the other hand, by the definition, $b_\alpha^i \in [t_\alpha^i]$ and $b_\beta^i \in [t_\beta^i]$. Since $[t_\alpha]$ and $[t_\beta]$ are comparable and $\beta < \alpha$, we need the following two cases.

Case 1. $[t_\alpha] \subseteq [t_\beta]$.

Since $[t_\alpha] \subseteq [t_{\beta+1}] = [t_\beta^i] \subseteq [t_\beta]$ for $i = 0$ or 1, without loss of generality, we may assume that $i = 0$. Then, $[t_{\beta+1}] = [t_\beta^0]$ and $[t_\beta^1] = [t_\beta] \setminus [t_{\beta+1}]$. It follows that $[t_\beta] \setminus [t_{\beta+1}] \in \mathcal{P}_\alpha \cap \mathcal{P}_\beta$. Since

$$(b_\alpha^1, b_\beta^1) \in [t_\alpha^1] \times [t_\beta^1] = [t_\alpha^1] \times ([t_\beta] \setminus [t_{\beta+1}]),$$

we conclude that $(b_\alpha^1, b_\beta^1) \notin U_\alpha \cup U_\beta$, but this is a contradiction.

Case 2. $[t_\alpha] \cap [t_\beta] = \emptyset$.

In this case, we can choose $\gamma < \beta$ and $i \in \{0, 1\}$ such that $[t_\alpha] \subseteq [t_\gamma^i]$ and $[t_\beta] \subseteq [t_\gamma^{1-i}]$. Hence, $\mathcal{P}_\alpha \ni [s_\gamma(\alpha)] \setminus [s_{\gamma+1}(\alpha)] = [t_\gamma^i]$ and $\mathcal{P}_\beta \ni [s_\gamma(\beta)] \setminus [s_{\gamma+1}(\beta)] = [t_\gamma^{1-i}]$. By the definitions of U_α and U_β, it follows that

$$(U_\alpha \cup U_\beta) \cap X \times X = [t_\gamma^i]^2 \cup [t_\gamma^{1-i}]^2 \cup \bigcup \{([s_\delta(\alpha)] \setminus [s_{\delta+1}(\alpha)])^2 : \delta < \gamma\},$$

because $s_\delta(\alpha) = s_\delta(\beta)$ if $\delta < \gamma$. On the other hand, since $b_\alpha^0 \in [t_\alpha^0] \subseteq [t_\gamma^i]$ and $b_\beta^0 \in [t_\beta^0] \subseteq [t_\gamma^{1-i}]$, $(b_\alpha^0, b_\beta^0) \in [t_\gamma^i] \times [t_\gamma^{1-i}]$. Thus, it follows that $(b_\alpha^0, b_\beta^0) \notin U_\alpha \cup U_\beta$, and which is a contradiction.

Since we have contradictions in both of the above cases, we can conclude that $g_\alpha G(U_\alpha) \cap g_\beta G(U_\beta) = \emptyset$. Therefore, $F(X)$ is not c.c.c. $\square$

Corollary 3.4 *Assume* ($\neg$SH). *Then there is a c.c.c. space X such that neither $F(X)$ nor $A(X)$ is c.c.c.*

Theorem 3.5 *The statement "the free (abelian) topological groups on c.c.c. space are c.c.c." is consistent with* ZFC.

We conclude this note with the following question.

Question *Is the free (abelian) topological group on a Suslin line c.c.c. ?*

References

[1] A. V. Arhangel'skiĭ, *Algebraic objects generated by topological structure*, J. Soviet Math. **45** (1989) 956-978.

[2] R. Engelking, *General Topology* (Heldermann, Berlin, 1989).

[3] M. I. Graev, *Free topological groups*, Izv. Akad. Nauk SSSR Ser. Mat. **12** (1948) 279-324 (in Russian); Amer. Math. Soc. Transl. **8** (1962) 305-364.

[4] E. Hewitt and K. Ross, *Abstract harmonic analysis I*, Academic Press, New York, (1963).

[5] A. A. Markov, *On free topological groups*, Izv. Akad. Nauk SSSR Ser. Mat. **9** (1945) 3-64 (in Russian); Amer. Math. Soc. Transl. **8** (1962) 195-272.

[6] V. G. Pestov, *Neighborhoods of the identity in free topological groups*, Herald Moscow State Univ. Ser. Math. Mech. **3** (1985) 8-10 (in Russian).

[7] M. G. Tkačenko, *On Suslin property in free topological groups over bicompacta*, Mat. Zametki **34** (1983) 601-607.

[8] M. G. Tkačenko, *On completeness of free Abelian topological groups*, Soviet Math. Dokl. **27** (1983), 341-345.

[9] M. G. Tkačenko, *On topologies of free groups*, Czech. Math. J. **34** (1984) 541-551.

[10] K. Yamada, *Metrizable subspaces of free topological groups on metrizable spaces*, to appear in Topology Proceedings.

Department of Mathematics, Faculty of Education, Shizuoka University, Shizuoka, 422 Japan

e-mail address: eckyama@ipc.shizuoka.ac.jp